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DESIGN FOR TWO LOADING CONDITIONS

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Abstract—The problem of the plastic design of a statically indeterminate truss with fixed geometry and connectivity subjected to two independent loading conditions is shown to decompose into two uncoupled single loading condition problems. The realizability of the resulting force system on a elastic truss is also discussed.

INTRODUCTION

THIS paper is concerned with the optimum design of a statically indeterminate truss with fixed connectivity and geometry which is subjected to two sets of loads. That is, given a truss configuration and its loads, find a set of bar areas which correspond to a minimum weight design. It may be added that while this is one of the most elementary design problems, it is not trivial since in the absence of experience or intuition the designer may begin with an unnecessarily large number of bars from which the optimal structure is to be selected.

With regard to the case of a single loading condition the situation is fairly well settled. For some time it has been known that design in this case is a linear programming problem and that the optimal structure is statically determinate [1]; it was shown recently [2, 3] that the commonly used iterative design procedure converges globally to a minimum weight design. Since the optimal structure is statically determinate, there is no question of realizability and it is immaterial to the optimal layout whether an elastic or a plastic design is being attempted.

The case of two loading conditions is more difficult. For plastic design, a system of bar forces which corresponds to a minimum weight structure is generated in the work which follows by decomposing the two loading condition problems into two, independent single loading condition problems which can be solved by available methods. The question of realizability is then discussed and it is seen that it is not always possible to realize this force system on an elastic structure. But in any case the plastic design serves as a lower bound for the weight of an elastic design.

THE DUAL LINEAR PROGRAMMING PROBLEM

For the truss subjected to two loading conditions, a dual linear programming problem has been formulated [4]:
primal problem

$$\text{minimize } \varphi = \frac{\sigma^a}{E}\sum_i L_i \max\{|F_i^1|, |F_i^2|\} \tag{1}$$

subject to $\tilde{N}F^1 = P^1$ and $\tilde{N}F^2 = P^2$
dual problem

$$\text{maximize } \psi = \tilde{P}^1\delta^1 + \tilde{P}^2\delta^2 \tag{2}$$

subject to $|N\delta^1| + |N\delta^2| \le (\sigma^a/E)L$.
Briefly, in the primal two sets of joint loads P^1 and P^2 are given and it is desired to find two sets of bar forces F^1 and F^2 which satisfy joint equilibrium and minimize the weight of the structure which is proportional to φ; in the dual problem it is desired to find two sets of joint displacements δ^1 and δ^2 which maximize ψ which is proportional to the work done by the external loads while keeping the absolute value of the sum of the member length changes less than the "allowable length change". In equations (1) and (2) the superscripts refer to loading conditions and it is assumed that the matrix N which appears in both the equilibrium equation

$$\tilde{N}F = P \tag{3}$$

and the member displacement–joint displacement equation

$$\Delta = N\delta \tag{4}$$

is given. Also known are the constants σ^a and E and the member lengths L_i.

This dual linear programming problem may be regarded as either an elastic design problem or a plastic design problem. From the point of view of plastic design the unknown force systems F^1 and F^2 must satisfy equilibrium and the safety requirement that A_i, the area of bar i, be large enough to carry the maximum force which occurs without yielding, i.e.

$$A_i \ge (\text{yield stress})^{-1} \times \max\{|F_i^1|, |F_i^2|\}. \tag{5}$$

From the point of view of elastic design (which is basically a nonlinear programming problem) the dual linear programming problem can be obtained by embedding the usual formulation in a system in which the constitutive equations are relaxed. This leads to the more simple linear programming formulation but requires the eventual consideration of the question of realizability. In this context σ^a and E can be regarded as the (constant) "allowable stress" and Young's modulus respectively and fully stressed design in which

$$A_i = (\text{allowable stress})^{-1} \max\{|F_i^1|, |F_i^2|\} \tag{6}$$

is implied.

THE DECOMPOSITION

In this section the dual linear programming problem, equations (1) and (2), is decomposed into two independent problems, each of which corresponds to a design for a single loading condition. In order to achieve this end it is only necessary to use the following relationships:

$$|x| + |y| \le 1 \Leftrightarrow \begin{matrix} |x+y| \le 1 \\ |x-y| \le 1 \end{matrix} \tag{7}$$

and

$$\max\{|x|, |y|\} = \tfrac{1}{2}|x+y| + \tfrac{1}{2}|x-y|. \tag{8}$$

Introducing these relationships into equations (1) and (2) yields after some recombination of terms:
primal problem

$$\text{minimize } \varphi = \frac{\sigma^a}{E}\sum_i L_i\{\tfrac{1}{2}|F_i^1 + F_i^2| + \tfrac{1}{2}|F_i^1 - F_i^2|\}$$

subject to

$$\tilde{N}\left(\frac{F^1+F^2}{2}\right) = \frac{P^1+P^2}{2}$$
$$\tilde{N}\left(\frac{F^1-F^2}{2}\right) = \frac{P^1-P^2}{2} \tag{9}$$

dual problem

$$\text{maximize } \psi = \left(\frac{\tilde{P}^1+\tilde{P}^2}{2}\right)(\delta^1+\delta^2) + \left(\frac{\tilde{P}^1-\tilde{P}^2}{2}\right)(\delta^1-\delta^2)$$

subject to

$$|N(\delta^1+\delta^2)| \le \frac{\sigma^a}{E}L$$
$$|N(\delta^1-\delta^2)| \le \frac{\sigma^a}{E}L. \tag{10}$$

It is convenient now to introduce new "sum" and "difference" variables

$$F^S = \tfrac{1}{2}(F^1+F^2) \qquad F^D = \tfrac{1}{2}(F^1-F^2)$$
$$P^S = \tfrac{1}{2}(P^1+P^2) \qquad P^D = \tfrac{1}{2}(P^1-P^2)$$
$$\delta^S = \delta^1+\delta^2 \qquad \delta^D = \delta^1-\delta^2.$$

Since the sum and difference variables are independent, the dual linear programming problem equations (9) and (10) in which the two loading conditions are coupled decomposes into two uncoupled problems in terms of the new variables:
sum problem

$$\text{minimize } \varphi = \frac{\sigma^a}{E}|\tilde{F}^S|L \quad \text{subject to } \tilde{N}F^S = P^S$$
$$\text{maximize } \psi^S = \tilde{P}^S\tilde{\delta} \quad \text{subject to } |N\delta^S| \le \frac{\sigma^a}{E}L \tag{11}$$

difference problem

$$\text{minimize } \varphi^D = \frac{\sigma^a}{E}|\tilde{F}^D|L \quad \text{subject to } \tilde{N}F^D = P^D$$
$$\text{maximize } \psi^D = \tilde{P}^D\delta^D \quad \text{subject to } |N\delta^D| \le \frac{\sigma^a}{E}L. \tag{12}$$

Both the sum and the difference problem have the form of a single loading condition problem and it is therefore possible to use methods already available for their solution. In particular, iterative design is known to converge to a minimum weight structure and appears to be easier to use than the simplex method.

Figure 1 shows a simple 3-bar truss which is used here to illustrate the decomposition just described. In this case

$$P^1 = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 \\ 1\frac{1}{2} \end{bmatrix} \quad P^S = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad P^D = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{bmatrix}$$

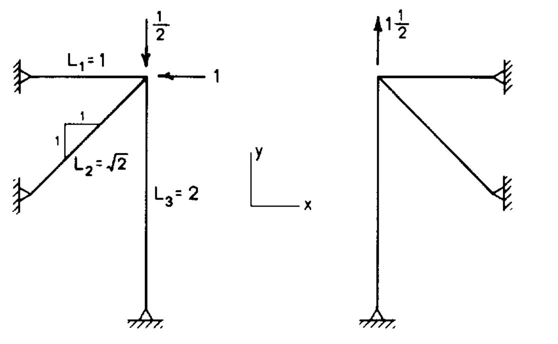

FIG. 1.

and the optimal forces are

$$F^S = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \quad F^D = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Let $\sigma^a = E = 1$ for simplicity. The optimal areas are then obtained by adding the absolute values of the components of F^S and F^D, i.e.

$$A_1 = \tfrac{1}{2} \qquad A_2 = 1/\sqrt{2} \qquad A_3 = 1$$

for an optimal weight of $3\frac{1}{2}$.

REALIZABILITY

With the exception of questions of shakedown, the plastic design is now complete. On the other hand, it is not at all clear that an elastic structure can be built, using the prescribed areas, which will develop the optimal forces or even an "allowable" set of forces under the prescribed loads; in fact, as a rule, this is not possible. The difficulty lies in the fact that while the sum and difference solutions (structures) are statically determinate, the structure which results from their combination is not and the load redistribution which occurs under the "addition" of two designs which satisfy allowable stress design criteria may cause overstress in the resulting design.

In order to achieve the greatest flexibility in discussing realizability for elastic structures it is desirable to introduce the concept of "hybrid action" [1] which is simply the advantageous use of "lack of fit". In this case the equations of a truss have the form

$$\begin{aligned} \tilde{N}F &= P \text{ (equilibrium)} \\ F &= K(\Delta - D) \text{ (constitutive equation)} \\ \Delta &= N\delta \text{ (member displacement–joint displacement)} \end{aligned}$$

in which the matrix D describes the bar length changes (lack of fit) under zero load. Since the optimal design procedure gives bar areas, the stiffness matrix K may be considered known. The question of realizability is then whether or not it is possible to find a lack of fit quantity D_i for each bar so that under both loadings P^1 and P^2 the stresses are within the allowable as they are when P^S and P^D act upon the sum and difference structures respectively. This is precisely the shakedown problem of plastic analysis [5], i.e. if there exists a matrix D with which the stresses remain less than the allowable for an elastic design, there also exists a set of residual forces under which the structure behaves elastically while going from load P^1 to load P^2 in an elastic–plastic design. Just as a structure may not shakedown, an optimal force system F^1, F^2 may not be realizable using the associated bar areas. In fact, the optimal force system given in the example is not realizable.

ITERATIVE PROCEDURES

It remains to discuss what to do when an optimal force system is not realizable. Motivated by the common iterative procedure for the case of a single loading condition,

$$A^{(n+1)} = \frac{|F^{(n)}|}{\sigma^a} \quad \text{or } A_i^{(n+1)} = \frac{|F_i^{(n)}|}{\sigma^a} \tag{13}$$

in which

$$F^{(n)} = K^{(n)}N(\tilde{N}K^{(n)}N)^{-1}P$$

it is most natural in the case of two loading conditions to iterate

$$A^{(n+1)} = \frac{\max\{|F^1|, |F^2|\}^{(n)}}{\sigma^a} = \frac{\{|F^S|+|F^D|\}^{(n)}}{\sigma^a} \tag{14}$$

in which

$$\begin{aligned} (F^1)^{(n)} &= K^{(n)}\{N[\tilde{N}K^{(n)}N]^{-1}[P^1+\tilde{N}K^{(n)}D^{(n)}]-D^{(n)}\} \\ (F^2)^{(n)} &= K^{(n)}\{N[\tilde{N}K^{(n)}N]^{-1}[P^2+\tilde{N}K^{(n)}D^{(n)}]-D^{(n)}\} \\ D^{(n)} &= -(K^{(n)})^{-1}(F^S)^{(n-1)}+(\Delta^S)^{n-1} \end{aligned}$$

starting with the optimal solution. With the exception of the use of the matrix D, equation (14) is a rather obvious generalization of equation (13). This use of D is motivated by the fact that a real structure has a single K matrix and a single D matrix which implies that under two loading conditions

$$F^1 = K(\Delta^1 - D)$$
$$F^2 = K(\Delta^2 - D) \tag{15}$$

and that the difference force

$$F^D = \tfrac{1}{2}(F' - F^2) = K\Delta^D \tag{16}$$

is independent of D. It is further motivated by the fact that equation (13) implies constant bar forces during the adjustment of the areas. The generalization of equation (13) to include hybrid action results in the last of equations (14).

There is an easier iteration scheme which produces identical results. It is based on the fact that using lack of fit or hybrid action it is possible to realize any single load force system, given K, by selecting the proper values for the matrix D. Assume that the optimal values of F^S and F^D, say $\bar{F}^S$ and $\bar{F}^D$, have been computed. Since F^D is independent of D [equation (16)] and in view of the preceding remark, for any value of K it is always possible to realize $\bar{F}^S$ using the appropriate D. This motivates a simplified version of equation (14),

$$A^{(n+1)} = \frac{|\bar{F}^S| + |F^D|^{(n)}}{\sigma^a} \tag{17}$$

in which

$$(F^D)^{(n)} = K^{(n)}N(\tilde{N}K^{(n)}N)^{-1}P^D.$$

If the iterative scheme is terminated with $n = N$, D can then be computed by solving

$$\bar{F}^S = K^{(N)}[(\Delta^D)^{(N)} - D]. \tag{18}$$

That the two schemes are equivalent can be verified by noting that starting from the optimal force system,

$$\bar{F}^S \equiv (F^S)^{(1)} = (F^S)^{(2)} = \cdots = (F^S)^{(N)}$$

using the first procedure. Figure 2 shows an application of equation (17) to the example problem used above. In this case the iterations converge to a statically determinate realizable solution with a weight of 4.

CONCLUDING REMARKS

The work just presented provides a direct means for obtaining a lower bound on minimum weight design when there are two loading conditions. If the force system which results is realizable, or if the structure shakes down, the force system is not just a lower bound but also an optimal solution. The shakedown theorem of plastic analysis can in fact be used to find the necessary lack of fit quantities. But beyond this, design for two loading conditions is conceptually difficult. It is in the understanding of this situation that the decomposition presented here should be particularly useful.

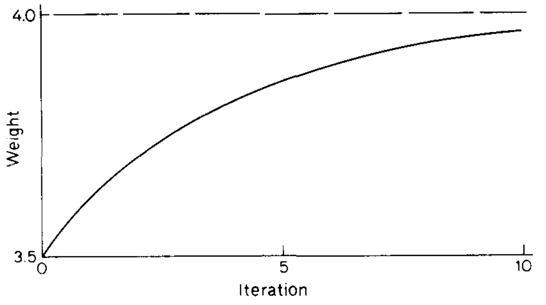

FIG. 2.

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Абстракт—Показано, что задача расчета в пластической области, статически неорределённой формы, с определённой геометрией и связностью, подверженной действию двух независимых условий нагрузки, распадается на две несвязанные задачи условия единичной нагрузки. Обсуждаеся, также, реализуемость системы суммарных сил, действующих на упругую ферму.